EQUIVALENCE OF TWO APPROACHES TO THE MKDV HIERARCHIES

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Abstract. The equivalence between the approaches of Drinfeld-Sokolov and Feigin-Frenkel to the mKdV hierarchies is established. A new derivation of the mKdV equations in the zero curvature form is given. Connection with the Baker-Akhiezer function and the tau-function is also discussed.

1. Introduction.

To each affine Kac-Moody algebra $\mathfrak{g}$ one can associate a modified Korteweg-de Vries (mKdV) hierarchy of non-linear partial differential equations. The mKdV hierarchy, which can be viewed as a refined form of a generalized KdV hierarchy (see [3]), is a completely integrable hamiltonian system. The equations of the hierarchy can be written in hamiltonian form, and the corresponding hamiltonian flows commute with each other.

It is known that the equations of an mKdV hierarchy can be represented in the zero curvature form

$$[\partial_{t_n} + V_n, \partial_z + V] = 0, \tag{1}$$

where t_n's are the times of the hierarchy, and $t_1 = z$. Here V and V_n are certain time dependent elements of the centerless affine algebra $\mathfrak{g}$. To write V explicitly, consider the principal abelian subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ (the precise definition is given below). It has a basis $p_i, i \in \pm I$, I being the set of all exponents of $\mathfrak{g}$ modulo the Coxeter number. Then

$$V = p_{-1} + \mathbf{u}(z),$$

where $\mathbf{u}(z)$ lies in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

The element p_{-1} has degree -1 with respect to the principal gradation of $\mathfrak{g}$, while $\mathbf{u}$ has degree 0. This makes finding an element V_n that satisfies (1) a non-trivial problem. Indeed, equation (1) can be written as

$$\partial_{t_n}\mathbf{u} = [\partial_z + p_{-1} + \mathbf{u}, V_n]. \tag{2}$$

The left hand side of (2) has degree 0. Therefore V_n should be such that the expression in the right hand side of (2) is concentrated in degree 0.

Such elements can be constructed by the following trick (see [17, 3, 15]). Suppose we found some $\mathcal{V}_n \in \mathfrak{g}$ which satisfies

$$[\partial_z + p_{-1} + \mathbf{u}, \mathcal{V}] = 0. \tag{3}$$

We can split $\mathcal{V}_n$ into the sum $\mathcal{V}_+ + \mathcal{V}_-$ of its components of positive and non-positive degrees with respect to the principal gradation. Then $V_n = \mathcal{V}_-$ has the property that the right hand side of (2) has degree 0. Indeed, from (3) we find

$$[\partial_z + p_{-1} + \mathbf{u}(z), \mathcal{V}_-] = -[\partial_z + p_{-1} + \mathbf{u}(z), \mathcal{V}_+],$$

which means that both commutators have neither positive nor negative homogeneous components. Therefore equation (2) makes sense. Now we have to find solutions of equation (3).

Drinfeld and Sokolov [3] proposed a powerful method of finding solutions of (3), which is closely related to the dressing method of Zakharov and Shabat [17]. Another approach was proposed by Wilson [15] (see also [13]).

Let us briefly explain the Drinfeld-Sokolov method. Recall that $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{b}_-$, where $\mathfrak{n}_+$ is the nilpotent subalgebra of $\mathfrak{g}$. Let N_+ be the corresponding Lie group. In [3] it was proved that there exists an N_+–valued function $M(z)$, which is called the dressing operator, such that

$$M(z)^{-1} \left(\partial_z + p_{-1} + \mathbf{u}(z)\right) M(z) = \partial_z + p_{-1} + \sum_{i \in I} h_i(z) p_i,$$

where h_i's are certain functions.

The dressing operator $M(z)$ is defined up to right multiplication by a z–dependent element of the subgroup A_+ of N_+ corresponding to the Lie algebra $\mathfrak{a}_+ = \mathfrak{a} \cap \mathfrak{n}_+$. Thus, $M(z)$ represents a coset in N_+/A_+. The element $\mathcal{V}_n = M(z) p_{-n} M(z)^{-1}$ clearly satisfies (3) and by substituting $V_n = (M(z) p_{-n} M(z)^{-1})_-$ in equation (1) for $n \in I$ one obtains the mKdV hierarchy.

Recently, another approach to mKdV hierarchies was proposed by Feigin and one of the authors [6, 7]. In this approach, the flows of the mKdV hierarchy are considered as vector fields on the space of jets of the function $\mathbf{u}(z)$. Let $\pi_0 = \mathbb{C}[u_i^{(n)}]_{i=1,\dots,l; n \geq 0}$, where $u_i = (\alpha_i, \mathbf{u})$ and $u_i^n = \partial_z^n u_i$, be the ring of differential polynomials in u_i's. In [7], π_0 was identified with the ring of algebraic functions on the homogeneous space N_+/A_+. Thus, each function $\mathbf{u}(z)$ gives rise to a function $K(z)$ with values in N_+/A_+. The Lie algebra $\mathfrak{a}_- = \mathfrak{a} \cap \mathfrak{b}_-$ naturally acts on N_+/A_+ from the right. Consider the derivation ∂_n on π_0 which corresponds to the infinitesimal action of p_{-n} on N_+/A_+. These derivations clearly commute with each other. Moreover, it was shown in [7] that ∂_1 coincides with ∂_z and therefore ∂_n's are evolutionary (i.e. commuting with ∂_z) derivations.

In this work we prove that the cosets $M(z)$ and $K(z)$, obtained by the constructions of [3] and [7], coincide. We then show that the derivation ∂_n satisfies equation (1) with $V_n = (K(z)p_{-n}K(z)^{-1})_-$. Thus, we establish an equivalence between the two constructions. Note that another approach to establishing this equivalence in the case of $\widehat{\mathfrak{sl}}_2$ based on KdV gauge fixing [3] was proposed by one of the authors in [4].

This gives us a direct identification of the flows corresponding to ∂_n and ∂_{t_n}.[1] Thus we obtain a new derivation of the zero curvature representation of the mKdV hierarchies.

We remark that there exist generalizations of the mKdV hierarchies which are associated to abelian subalgebras of $\mathfrak{g}$ other then $\mathfrak{a}$. It is known that the Drinfeld-Sokolov approach can be applied to these generalized hierarchies [1, 9]. On the other hand, the approach of [7] can also be applied; in the case of the non-linear Schrödinger hierarchy, which corresponds to the homogeneous abelian subalgebra of $\mathfrak{g}$, this has been done by Feigin and one of the authors [8]. The results of our paper can be extended to establish the equivalence between the two approaches in this general context.

The paper is arranged as follows. In Sect. 2 we recall the construction of [7] and derive the zero curvature equations. In Sect. 3 we prove that the cosets M and K coincide and that the derivations ∂_n and ∂_{t_n} coincide. We also explain the connection with the KdV hierarchies. In Sect. 4 we construct a natural system of coordinates on the group N_+ and using it give another proof of the equivalence of two formalisms. Finally, in Sect. 5 we obtain explicit formulas for the one-cocycles defined in [7] and the densities of the hamiltonians of the mKdV hierarchy. We also discuss a connection between the formalism of [7] and τ–functions.

2. Unipotent cosets.

2.1. Notation. Let $\widetilde{\mathfrak{g}}$ be an affine algebra. It has generators $e_i, f_i, \alpha_i^\vee, i = 0, \dots, l$, and d, which satisfy the standard relations [10]. The Lie algebra $\widetilde{\mathfrak{g}}$ carries a non-degenerate invariant inner product $(\cdot,\cdot)$. One associates to $\widetilde{\mathfrak{g}}$ the labels $a_i, a_i^\vee, i = 0, \dots, l$, the exponents $d_i, i = 1, \dots, l$, and the Coxeter number h, see [10]. We denote by I the set of all positive integers, which are congruent to the exponents of $\widetilde{\mathfrak{g}}$ modulo h (with multiplicities). The elements $e_i, i = 0, \dots, l$, and $f_i, i = 0, \dots, l$, generate the nilpotent subalgebras $\mathfrak{n}_+$ and $\mathfrak{n}_-$ of $\widetilde{\mathfrak{g}}$, respectively. The elements $\alpha_i^\vee$ generate the Cartan subalgebra $\widetilde{\mathfrak{h}}$ of $\widetilde{\mathfrak{g}}$. We have: $\widetilde{\mathfrak{g}} = \mathfrak{n}_+ \oplus \mathfrak{b}_-$, where $\mathfrak{b}_- = \mathfrak{n}_- \oplus \widetilde{\mathfrak{h}}$. Each $x \in \widetilde{\mathfrak{g}}$ can be uniquely written as $x_+ + x_-$, where $x_+ \in \mathfrak{n}_+$ and $x_- \in \mathfrak{b}_-$.

The element $C = \sum_{i=0}^{l} a_i^\vee \alpha_i^\vee$ of $\widetilde{\mathfrak{h}}$ is a central element of $\widetilde{\mathfrak{g}}$. Let $\mathfrak{g}$ be the quotient of

[1] In [7] the following indirect proof of this fact was given: the derivations ∂_n were identified in [7] with the symmetries of the affine Toda equation corresponding to $\mathfrak{g}$. But it is known that mKdV equations constitute all symmetries of the affine Toda equation, see [3, 13, 15].

$[\widetilde{\mathfrak{g}}, \widetilde{\mathfrak{g}}]$ by $\mathbb{C}C$. We identify $\widetilde{\mathfrak{g}}$ with the direct sum $\mathfrak{g} \oplus \mathbb{C}C \oplus \mathbb{C}d$. The Lie algebra $\mathfrak{g}$ has a Cartan decomposition $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$, where $\mathfrak{h}$ is spanned by $\alpha_i^\vee, i = 1, \ldots, l$.

Set $p_1 = \sum_{i=0}^{l} a_i e_i$. Let $\mathfrak{a}$ be the centralizer of p_1 in $\mathfrak{g}$. This is an abelian subalgebra of $\mathfrak{g}$ which we call the principal abelian subalgebra. We have a decomposition: $\mathfrak{a} = \mathfrak{a}_+ \oplus \mathfrak{a}_-$, where $\mathfrak{a}_+ = \mathfrak{a} \cap \mathfrak{n}_+$, and $\mathfrak{a}_- = \mathfrak{a} \cap \mathfrak{b}_-$. It is known that $\mathfrak{a}_\pm$ is spanned by elements $p_i, i \in \pm I$, which have degrees $\deg p_i = i$ with respect to the principal gradation of $\mathfrak{g}$. In particular, we can choose $p_{-1} = \sum_{i=0}^{l} \frac{(\alpha_i, \alpha_i)}{2} f_i$, where α_i's are the simple roots of $\mathfrak{g}$, considered as elements of $\mathfrak{h}$ using the inner product.

Remark 1. For all affine algebras except $D_{2n}^{(1)}$, each exponent occurs exactly once. In the case of $D_{2n}^{(1)}$, the exponent $2n-1$ has multiplicity 2. In this case, there are two generators of $\mathfrak{a}$, p_i^1 and p_i^2, for i congruent to $2n-1$ modulo the Coxeter number $4n-2$. □

Let N_+ be the Lie group of $\mathfrak{n}_+$. This is a prounipotent proalgebraic group (see, e.g., [7]). The exponential map $\exp : \mathfrak{n}_+ \to N_+$ is an isomorphism of proalgebraic varieties. Let A_+ be the image of $\mathfrak{a}_+$ under this map. The Lie algebra $\mathfrak{g}$ acts on N_+ from the right because N_+ can be embedded as an open subset in the flag manifold $B_-\backslash G$ of $\mathfrak{g}$. Therefore the normalizer of $\mathfrak{a}_+$ in $\mathfrak{g}$ acts on N_+/A_+ from the right. In particular, $\mathfrak{a}_-$ acts on N_+/A_+, and each $p_{-n}, n \in I$ gives rise to a derivation of $\mathbb{C}[N_+/A_+]$, see [7]. These derivations commute with each other.

2.2. Actions on the space of jets. Consider the space U of jets of a smooth function $\mathbf{u}(z) : \mathbb{A}^1 \to \mathfrak{h}$. The space U is the inverse limit of the finite-dimensional vector spaces $U_N = \operatorname{span}\{u_i^{(n)}\}_{i=1,\ldots,l;n=1,\ldots,N}$, where $u_i = (\alpha_i, \mathbf{u})$, and $u_i^{(n)} = \partial_z^n u_i$. Thus, the ring π_0 of regular functions on U is $\mathbb{C}[u_i^{(n)}]_{i=1,\ldots,l;n\geq 0}$. The derivative ∂_z gives rise to a derivation of π_0.

Theorem 1 ([7], Theorem 2). *There is an isomorphism of rings*

$$\mathbb{C}[N_+/A_+] \simeq \mathbb{C}[u_i^{(n)}],$$

under which p_{-1} gets identified with ∂_z.

Let ∂_n be the derivation of $\mathbb{C}[u_i^{(n)}]$ corresponding to p_{-n} under this isomorphism. The theorem shows that the derivations ∂_n are evolutionary, i.e. commuting with ∂_z. We would like to represent the action of these derivations on $\mathbf{u}(z)$ explicitly in the zero curvature form (1).

For $g \in G$ and $x \in \mathfrak{g}$ we will write gxg^{-1} for $\mathrm{Ad}_g(x)$.

Proposition 1. *For $K \in N_+/A_+$,*

$$[\partial_m + (Kp_{-m}K^{-1})_-, \partial_n + (Kp_{-n}K^{-1})_-] = 0, \qquad \forall m, n \in I. \tag{4}$$

Let us explain the meaning of formula (4). For each $K \in N_+/A_+$, $Kp_{-n}K^{-1}$ is a well-defined element of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ can be realized as $\overline{\mathfrak{g}} \otimes \mathbb{C}((t))$ (or a subalgebra thereof if $\mathfrak{g}$ is twisted) for an appropriate finite-dimensional Lie algebra $\overline{\mathfrak{g}}$. If we choose a basis in $\overline{\mathfrak{g}}$, we can consider an element of $\mathfrak{g}$ as a set of Laurent power series. In particular, for $Kp_{-n}K^{-1}$, any Fourier coefficient of each of these power series gives us an algebraic function on N_+/A_+. Hence, by Theorem 1, each coefficient corresponds to a differential polynomial in u_i's, and we can apply ∂_m to it.

In order to prove formula (4), we need to find an explicit formula for the action of ∂_n on $Kp_{-m}K^{-1}$.

Let us first obtain a formula for the infinitesimal action of an element of $\mathfrak{g}$ on N_+. Recall from [7] that since N_+ embeds as an open subset in the flag manifold $B_-\backslash G$, the Lie algebra $\mathfrak{g}$ infinitesimally acts on N_+ from the right by vector fields. Therefore $\mathfrak{g}$ acts by derivations on $\mathbb{C}[N_+]$. Since N_+ acts on $\mathfrak{g}$, we obtain a homomorphism from N_+ to the group of automorphisms of $\overline{\mathfrak{g}}$ over the ring $\mathbb{C}[[t]]$.

Now we can consider each element of N_+ as a matrix, whose entries are Taylor power series. Each Fourier coefficient of such a series defines an algebraic function on N_+, and the ring $\mathbb{C}[N_+]$ is generated by these functions. Hence any derivation of $\mathbb{C}[N_+]$ is uniquely determined by its action on these functions. We can write this action concisely as follows: $\nu \cdot x = y$, where x is a "test" matrix representing an element of N_+, and y is another matrix, whose entries are the results of the action of ν on the entries of x.

For $a \in \mathfrak{g}$, let a^R be the derivation of $\mathbb{C}[N_+]$ corresponding to the right infinitesimal action of a on N_+. For $b \in \mathfrak{n}_+$, let b^L be the derivation of $\mathbb{C}[N_+]$ corresponding to the left infinitesimal action of b on N_+.

Lemma 1.

$$a^R \cdot x = (xax^{-1})_+ x, \qquad \forall a \in \mathfrak{g}, \tag{5}$$

$$b^L \cdot x = bx, \qquad \forall b \in \mathfrak{n}_+. \tag{6}$$

Proof. Consider a one-parameter subgroup $a(\epsilon)$ of G, such that $a(\epsilon) = 1 + \epsilon a + o(\epsilon)$. We have: $x \cdot a(\epsilon) = x + \epsilon xa + o(\epsilon)$. For small ϵ we can factor $x \cdot a(\epsilon)$ into a product $y_- y_+$, where $y_+ = x + \epsilon y_+^{(1)} + o(\epsilon) \in N_+$ and $y_- = 1 + \epsilon y_-^{(1)} \in B_-$. We then find that $y_-^{(1)} x + y_+^{(1)} = xa$, from which we conclude that $y_+^{(1)} = (xax^{-1})_+ x$. This proves formula (5). Formula (6) is obvious. □

It follows from formula (5) that

$$a^R \cdot xvx^{-1} = [(xax^{-1})_+, xvx^{-1}], \qquad a, v \in \mathfrak{g}. \tag{7}$$

If a and v are both elements of $\mathfrak{a}$, then formula (7) does not change if we multiply x from the right by an element of A_+. Denote by K the coset of x in N_+/A_+. Then

we can write:

$$\partial_n \cdot KvK^{-1} = [(Kp_{-n}K^{-1})_+, KvK^{-1}], \qquad v \in \mathfrak{a}. \tag{8}$$

Proof of Proposition 1. Substituting $v = p_{-m}$ into formula (8), we obtain:

$$\partial_n \cdot Kp_{-m}K^{-1} = [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}].$$

Hence

$$\partial_n \cdot (Kp_{-m}K^{-1})_- = [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- = [(Kp_{-n}K^{-1})_+, (Kp_{-m}K^{-1})_-]_-.$$

Therefore we obtain:

$$\begin{aligned}
&[\partial_m + (Kp_{-m}K^{-1})_-, \partial_n + (Kp_{-n}K^{-1})_-] \\
=&\partial_m \cdot (Kp_{-n}K^{-1})_- - \partial_n \cdot (Kp_{-n}K^{-1})_- + [(Kp_{-m}K^{-1})_-, (Kp_{-n}K^{-1})_-] \\
=&[(Kp_{-m}K^{-1})_+, (Kp_{-n}K^{-1})_-]_- - [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- \\
+&[(Kp_{-m}K^{-1})_-, (Kp_{-n}K^{-1})_-]_-.
\end{aligned}$$

Adding up the first and the last terms, we obtain

$$\begin{aligned}
[Kp_{-m}K^{-1}, (Kp_{-n}K^{-1})_-]_- - [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]_- = \\
= [Kp_{-m}K^{-1}, Kp_{-n}K^{-1}]_- = 0,
\end{aligned}$$

and the proposition is proved. □

2.3. Zero curvature form. In order to write the action of ∂_n in the zero curvature form, we will use Proposition 1 in the case $m = 1$. But first we determine $(Kp_{-1}K^{-1})_-$ explicitly.

Lemma 2.

$$(Kp_{-1}K^{-1})_- = p_{-1} + \mathbf{u}. \tag{9}$$

It is clear that $(Kp_{-1}K^{-1})_- = p_{-1} + x$, where $x \in \mathfrak{h}$. Hence we need to show that $x = \mathbf{u}$, or, equivalently, that $(\alpha_i, x) = u_i, i = 1, \ldots, l$. We can rewrite the latter formula as $u_i = (\alpha_i, Kp_{-1}K^{-1})_-$, and hence as $u_i = (\alpha_i, Kp_{-1}K^{-1})$. To establish the last formula, recall the interpretation of u_i from [7].

Consider the module M_λ^* contragradient to the Verma module M_λ over $\mathfrak{g}$ with highest weight λ. This module can be realized in the space $\mathbb{C}[N_+]$ in such a way that the highest weight vector v_λ corresponds to the constant function. For $a \in \mathfrak{g}$ denote by $f_\lambda(a)$ the function on N_+ which corresponds to $a \cdot v_\lambda$. Then $u_i = f_{\alpha_i}(p_{-1})$ [7]. But in fact there is a general formula for $f_\lambda(a)$ due to Kostant [12].

Proposition 2 ([12]**, Theorem 2.2).** *Consider $\lambda \in \mathfrak{h}^*$ as a functional on $\mathfrak{g}$ which is trivial on $\mathfrak{n}_\pm$. Let $\langle \cdot, \cdot \rangle$ be the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$. Then $f_\lambda(a)(x) = \langle \lambda, xax^{-1} \rangle$.*

The formula above immediately implies that the function u_i on N_+/A_+ takes value $(\alpha_i, Kp_{-1}K^{-1})$ at $K \in N_+/A_+$. This completes the proof of Lemma 2.

Now specializing $m = 1$ in formula (4) and using Lemma 2 we obtain the zero curvature representation of the equations.

Theorem 2.

$$[\partial_z + p_{-1} + \mathbf{u}, \partial_n + (Kp_{-n}K^{-1})_-] = 0. \tag{10}$$

This equation can be rewritten as

$$\partial_n \mathbf{u} = \partial_z (Kp_{-n}K^{-1})_- + [p_{-1} + \mathbf{u}, (Kp_{-n}K^{-1})_-]. \tag{11}$$

The map $K \to Kp_{-n}K^{-1}$ defines an embedding of N_+/A_+ into $\mathfrak{g}$ as an N_+–orbit. The entries of the matrix $Kp_{-n}K^{-1}$ are Laurent series in t whose coefficients are differential polynomials in $u_i, i = 1, \dots, l$ (see the paragraph after Proposition 1). Equation (11) expresses $\partial_n u_i$ in terms of differential polynomials in u_i's. Since, by construction, ∂_n commutes with $\partial_1 \equiv \partial_z$, formula (11) uniquely determines ∂_n as an evolutionary derivation of $\mathbb{C}[u_i^{(n)}]$.

3. Equivalence with the Drinfeld-Sokolov formalism.

3.1. Identification of cosets. For $v \in \mathfrak{a}$ and $K \in N_+/A_+$ set

$$\mathcal{V}_v = KvK^{-1}. \tag{12}$$

Since $\mathfrak{a}$ is commutative, this is a well-defined element of $\mathfrak{g}$.

Proposition 3.

$$[\partial_z + p_{-1} + \mathbf{u}, \mathcal{V}_v] = 0, \qquad \forall v \in \mathfrak{a}. \tag{13}$$

Proof. Using formula (8) and Lemma 2 we obtain:

$$\partial_z \mathcal{V}_v = [(Kp_{-1}K^{-1})_+, \mathcal{V}_v] = -[(Kp_{-1}K^{-1})_-, \mathcal{V}_v] = -[p_{-1} + \mathbf{u}, \mathcal{V}_v]. \tag{14}$$

□

Now we define the Drinfeld-Sokolov dressing operator M.

Proposition 4 ([3], Proposition 6.2). *There exists an element $M = M(z) \in N_+$, such that*

$$M^{-1} (\partial_z + p_{-1} + \mathbf{u}(z)) M = \partial_z + p_{-1} + \sum_{i \in I} h_i p_i, \tag{15}$$

where h_i's are functions. M is defined uniquely up to right multiplication by a (possibly z–dependent) element of A_+. One can choose M in such a way that all entries of its matrix and all h_i's are polynomials in $u_i^{(n)}, i = 1, \dots, l; n \geq 0$.

The proposition defines a map from the space of smooth functions $\mathbf{u}(z) : \mathbb{A}^1 \to \mathfrak{h}$ to the space of smooth functions $\mathbb{A}^1 \to N_+/A_+$, $\mathbf{u}(z) \to M(z)$. On the other hand, Theorem 1 also defines such a map $\mathbf{u}(z) \to K(z)$. The following lemma will allow us to identify these two maps.

Remark 2. Note that both maps are local in the following sense. For each z, $M(z)$ and $K(z)$ depend only on the jet of $\mathbf{u}$ at z. In particular, for each $v \in \mathfrak{a}$, all entries of the matrices $M(z)vM(z)^{-1}$ and $K(z)vK(z)^{-1}$ are Taylor series whose coefficients are differential polynomials in u_i's. $\square$

Lemma 3 ([3]). *Let $\mathcal{V}$ be an element of $\mathfrak{g}$ of the form $\mathcal{V} = p_{-n}+$ terms of degree higher than $-n$ with respect to the principal gradation on $\mathfrak{g}$, such that*

$$[\partial_z + p_{-1} + \mathbf{u}, \mathcal{V}] = 0. \tag{16}$$

Then $\mathcal{V} = MvM^{-1}$, where $M \in N_+$ satisfies (15) *and $v \in \mathfrak{a}$ is such that $v = p_{-n}+$ terms of degree higher than $-n$.*

The proof of the lemma requires the following important result.

Proposition 5 ([11], **Proposition 3.8).** *The Lie algebra $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{a} \oplus \mathrm{Im}(\mathrm{ad}\, p_{-n})$ for each $n \in I$. Moreover,* $\mathrm{Ker}(\mathrm{ad}\, p_{-n}) = \mathfrak{a}$.

Proof of Lemma 3. If $\mathcal{V}$ satisfies (16), then we obtain from Proposition 4:

$$[\partial_z + p_{-1} + \sum_{i \in I} h_i p_i, M^{-1}\mathcal{V}M] = 0. \tag{17}$$

We can write $M^{-1}\mathcal{V}M$ as a sum $\sum_j v_j$ of its homogeneous components of principal degree j. According to Proposition 5, each v_j can be split into the sum of $v_j^0 \in \mathfrak{a}_+$ and $v_j^1 \in \mathrm{Im}(\mathrm{ad}\, p_{-1})$.

Suppose that $M^{-1}\mathcal{V}M$ does not lie in $\mathfrak{a}_+$. Let j_0 be the smallest number such that $v_{j_0}^1 \neq 0$. Then the term of smallest degree in (17) is $[p_{-1}, v_{j_0}^1]$ which is non-zero, because $\mathrm{Ker}(\mathrm{ad}\, p_{-n}) = \mathfrak{a}_+$. Hence (17) can not hold.

Therefore $M^{-1}\mathcal{V}M \in \mathfrak{a}$. But then (17) gives: $\partial_z v_j = 0$ for all j. This means that each v_j is a constant element of $\mathfrak{a}$, and Lemma follows. $\square$

Theorem 3. *The cosets $M(z)$ and $K(z)$ in N_+/A_+ assigned in* [3] *and* [7]*, respectively, to the jet of function $\mathbf{u} : \mathbb{A}^1 \to \mathfrak{h}$ at z, coincide.*

Proof. According to Proposition 3,

$$[\partial_z + p_{-1} + \mathbf{u}, K p_{-n} K^{-1}] = 0.$$

Since $Kp_{-n}K^{-1} = p_{-n}+$ terms of degree higher than $-n$ with respect to the principal gradation, we obtain from Lemma 3 that $Kp_{-n}K^{-1} = MvM^{-1}$, where $M \in N_+$ satisfies (15) and $v \in \mathfrak{a}$. This implies that $v = p_{-n}$ and that $M = K$ in N_+/A_+.

Indeed, from the equality

$$Kp_{-n}K^{-1} = MvM^{-1}$$

we obtain that $(M^{-1}K)p_{-n}(M^{-1}K)^{-1}$ lies in $\mathfrak{a}$. We can represent $M^{-1}K$ as $\exp y$ for some $y \in \mathfrak{n}_+$. Then $(M^{-1}K)p_{-n}(M^{-1}K)^{-1} = v$ can be expressed as a linear combination of multiple commutators of y and p_{-n}:

$$e^y p_{-n}(e^y)^{-1} = \sum_{n\geq 0} \frac{1}{n!}(\operatorname{ad} y)^n \cdot p_{-n}.$$

We can write $y = \sum_{j>0} y_j$, where y_j is the homogeneous component of y of principal degree j. It follows from Proposition 5 that $\mathfrak{n}_+ = \mathfrak{a}_+ \oplus \operatorname{Im}(\operatorname{ad} p_{-n})$. Therefore each y_j can be further split into a sum of $y_j^0 \in \mathfrak{a}_+$ and $y_j^1 \in \operatorname{Im}(\operatorname{ad} p_{-1})$.

Suppose that y does not lie in $\mathfrak{a}_+$. Let j_0 be the smallest number such that $y_{j_0}^1 \neq 0$. Then the term of smallest degree in $e^y p_{-n}(e^y)^{-1}$ is $[y_{j_0}^1, p_{-n}]$ which lies in $\operatorname{Im}(\operatorname{ad} p_{-n})$ and is non-zero, because $\operatorname{Ker}(\operatorname{ad} p_{-n}) = \mathfrak{a}_+$. Hence $e^y p_{-n}(e^y)^{-1}$ can not be an element of $\mathfrak{a}_+$.

Therefore $y \in \mathfrak{a}_+$ and so $M^{-1}K \in A_+$, which means that M and K represent the same coset in N_+/A_+, and that $v = p_{-n}$. $\square$

3.2. Identification of the equations. As was explained in the previous section, Theorem 1 allows us to define a set of commuting derivations $\partial_n, n \in I$, of π_0, or equivalently, vector fields on the space of jets U. These derivations can be represented in the zero curvature form (10).

On the other hand, in [3] another set of derivations $\partial_{t_n}, n \in I$, of π_0 was defined in the zero curvature form. Set

$$V_n = \left(M(z)p_{-n}M(z)^{-1}\right)_- , \tag{18}$$

where $M(z)$ is defined in Proposition 4. In particular, formula (15) shows that $V_1 = V = p_{-1} + \mathbf{u}$. The nth zero curvature equation is equation (1). Now we obtain from Theorem 3

Theorem 4. *The derivations ∂_n and ∂_{t_n} coincide.*

Remark 3. This theorem together with Theorem 1 implies that solutions of the mKdV hierarchy are just the integral curves of the vector fields of the infinitesimal action of the Lie algebra $\mathfrak{a}_-$ on N_+/A_+. $\square$

Remark 4. The variable t^{-1} appearing in the affine algebra $\mathfrak{g}$ is often denoted by λ, and is called the spectral parameter. $\square$

Remark 5. For each $n \in I \cup -I$, the map $K \mapsto Kp_{-n}K^{-1}$ defines an embedding $N_+/A_+ \to \mathfrak{g}$, because the stabilizer of p_{-n} in N_+ is A_+. In practice, it is convenient to find $Kp_{-n}K^{-1}$ using equation

$$[\partial_z + p_{-1} + \mathbf{u}(z), Kp_{-n}K^{-1}] = 0, \tag{19}$$

which follows from formula (13). We can split $Kp_{-n}K^{-1}$ into the sum of homogeneous components lying in $\mathfrak{a}$ and in $\mathrm{Im}(\mathrm{ad}\, p_{-1})$. These homogeneous components can then be determined recursively using equation (19) as explained in [15], Sect. 3.

This recursion is actually non-trivial: at certain steps one has to take the anti-derivative of a differential polynomial. But we know from Proposition 3 that the element $Kp_{-n}K^{-1}$ satisfies (19) and that its entries are differential polynomials (see the paragraph after Proposition 1). Therefore whenever an anti-derivative occurs, it can be resolved in the ring of differential polynomials. Another proof of this fact has been given by Wilson [15].

Every time we compute the anti-derivative, we have the freedom of adding an arbitrary constant. This corresponds to adding to $Kp_{-n}K^{-1}$ a linear combination of Kp_mK^{-1} with $m > -n$. $\square$

Remark 6. The map which attaches to u_i's a coset in N_+/A_+ can be viewed as a universal feature in various approaches to soliton equations. In this section we have explained how these maps arise in the formalisms of [3] and [7] and proved that these maps coincide.

But a map to N_+/A_+ can also be found, in a somewhat disguised form, in the approach to the soliton equations based on Sato's Grassmannian, see [14, 16]. One can associate to u_i's their Baker-Akhiezer function Ψ which is a solution of the equation

$$(\partial_z + p_{-1} + \mathbf{u}(z))\Psi = 0, \tag{20}$$

and more generally the equations

$$(\partial_n + (Kp_{-n}K^{-1})_-)\Psi = 0, \qquad \forall n \in I. \tag{21}$$

In our notation, Segal and Wilson [14, 16] attach in the case of $\mathfrak{g} = \mathfrak{sl}_n$ a Baker-Akhiezer function Ψ to each point x of the flag manifold $B_-\backslash G$ using its realization via an infinite Grassmannian. The flows of the mKdV hierarchy then correspond to the right infinitesimal action of $\mathfrak{a}_-$ on the flag manifold. As x moves along the integral curves of the vector fields ∂_n, the Baker-Akhiezer function evolves according to the mKdV hierarchy and so does the function $\mathbf{u}$. One shows [14] that Ψ is regular at a given set of times of the hierarchy if the corresponding point of the flag manifold lies in the big cell (which is isomorphic to N_+). Moreover, $\mathbf{u}$ does not change under the right action of A_+ on x [16]. Thus, one obtains a map which assigns to $\mathbf{u}$ an element of N_+/A_+.

In [16] the equivalence between the dressing method and the Grassmannian approach was established (see also [9]). Therefore this map coincides with the map studied in our paper.

We will derive an explicit formula for the Baker-Akhiezer function in Sect. 4. □

3.3. From mKdV to KdV. First let us recall the definition of the KdV hierarchies from [3]. Consider the operator

$$\partial_z + p_{-1} + \mathbf{u}(z), \tag{22}$$

where now $\mathbf{u}(z)$ lies in the finite-dimensional Borel subalgebra $\mathfrak{h} \oplus \overline{\mathfrak{n}}_+$, where $\overline{\mathfrak{n}}_+$ is generated by $e_i, i = 1, \dots, l$. Drinfeld and Sokolov construct in [3] the dressing operator and the zero-curvature equations (1) for this operator in the same way as for the mKdV hierarchy using formulas (15) and (18).

The Lie group $\overline{N}_+$ of $\overline{\mathfrak{n}}_+$ acts naturally on the space of operators (22) and these equations preverve the corresponding gauge equivalence classes [3]. Thus one obtains a system of compatible evolutionary equations on the gauge equivalence classes, which is called the generalized KdV hierarchy corresponding to $\widetilde{\mathfrak{g}}$. Let $\overline{\mathfrak{n}}^0_+$ be a subspace of $\overline{\mathfrak{n}}_+$ that is transversal to the image in $\overline{\mathfrak{n}}_+$ of the operator $\operatorname{ad} \overline{p}_{-1}$, where $\overline{p}_{-1} = \sum_{i=1}^{l} \frac{(\alpha_i,\alpha_i)}{2} f_i$. It is shown in [3] that each equivalence class contains a unique operator (22) satisfying the condition that $\mathbf{u} \in \overline{\mathfrak{n}}^0_+$.

The space $\overline{\mathfrak{n}}^0_+$ is l–dimensional. If we choose coordinates $v_1, \dots, v_l$ of $\overline{\mathfrak{n}}^0_+$, then the KdV equations can be written as partial differential equations on v_i's. On the other hand, the dressing operator $M(z)$ corresponding to a gauge class of operators (22) should now be considered as a double coset in $\overline{N}_+ \backslash N_+ / A_+$. Thus, a smooth function $\mathbf{v}(z) = (v_1(z), \dots, v_l(z)) : \mathbb{A}^1 \to \overline{\mathfrak{n}}^0_+$ gives rise to a smooth function $\mathbb{A}^1 \to \overline{N}_+ \backslash N_+ / A_+$.

Denote by $\mathcal{L}$ the space of all operators (22) where $\mathbf{u} \in \mathfrak{h}$, and by $\widetilde{\mathcal{L}}$ the space of all operators (22) where $\mathbf{u} \in \overline{\mathfrak{n}}^0_+$. We obtain a surjective map $\mathcal{L} \to \widetilde{\mathcal{L}}$, which sends an operator from $\mathcal{L}$ to the unique representative of its gauge class lying in $\widetilde{\mathcal{L}}$. This map is called the Miura transformation. It induces a homomorphism of differential rings $\mathbb{C}[v_i^{(n)}] \to \mathbb{C}[u_i^{(n)}]$.

It was shown in [6] that the image of $\mathbb{C}[v_i^{(n)}]$ in $\mathbb{C}[u_i^{(n)}]$ coincides with the invariant subspace of $\mathbb{C}[u_i^{(n)}]$ under the left action of the group $\overline{N}_+$. Hence we obtain from Theorem 1 that $\mathbb{C}[v_i^{(n)}] \simeq \mathbb{C}[\overline{N}_+ \backslash N_+ / A_+]$. Thus, we obtain a local map which assigns to each smooth function $\mathbf{v}(z) : \mathbb{A}^1 \to \overline{\mathfrak{n}}^0_+$ a smooth function $\mathbb{A}^1 \to \overline{N}_+ \backslash N_+ / A_+$. According to the results of this section, this map coincides with the Drinfeld-Sokolov map defined above. We also see that the KdV flows on $\overline{N}_+ \backslash N_+ / A_+]$ correspond to the right infinitesimal action of $\mathfrak{a}_-$ on it considered as an open subset of the "loop space" $\overline{G}[t^{-1}] \backslash G / A_+$. Thus, the passage from mKdV hierarchy to the KdV hierarchy simply consists of projecting from the flag manifold $B_- \backslash G$ to the loop space $\overline{G}[t^{-1}] \backslash G$.

Remark 7. Drinfeld and Sokolov attached in [3] a generalized KdV hierarchy to each vertex of the Dynkin diagram of $\widetilde{\mathfrak{g}}$; the hierarchy considered above corresponds to the 0th node. In general, we obtain the following picture.

Fix j between 0 and l. Let $\overline{\mathfrak{n}}^j_+$ be the finite-dimensional Lie subalgebra of $\mathfrak{n}_+$ generated by $e_i, i \neq j$. Let $\overline{N}^j_+$ be the corresponding Lie subgroup of N_+. The dressing operator of the jth generalized KdV hierarchy gives rise to a double coset in $\overline{N}^j_+ \backslash N_+/A_+$. On the other hand, there is an isomorphism between $\mathbb{C}[\overline{N}^j_+ \backslash N_+/A_+]$ and the ring of $\overline{\mathfrak{n}}^j_+$–invariants of $\mathbb{C}[u_i^{(n)}]$ (with respect to the left action). The latter is itself a ring of differential polynomials in l variables. Note that it coincides with the intersection of kernels of the operators $e_i^L, i \neq j$, which are classical limits of the so-called screening operators (see [6]). $\square$

4. Realization of $\mathbb{C}[N_+]$ as a polynomial ring.

The approach to the mKdV and affine Toda equations used in [7] and here is based on Theorem 1 which identifies $\mathbb{C}[N_+/A_+]$ with the ring of differential polynomials $\mathbb{C}[u_i^{(n)}]_{i=1,\dots,l;n\geq 0}$. In this section we add to the latter ring new variables corresponding to A_+ and show that the larger ring thus obtained is isomorphic to $\mathbb{C}[N_+]$. An analogous construction has been given in [5] in the lattice case.

4.1. Coordinates on N_+. Consider $u_i^{(n)}, i = 1, \dots, l; n \geq 0$, as A_+–invariant regular functions on N_+. Recall that

$$u_i(x) = (\alpha_i, x p_{-1} x^{-1}), \qquad x \in N_+,$$

Now choose an element χ of $\widetilde{\mathfrak{h}}$, such that $(\chi, C) \neq 0$. Introduce the regular functions $\chi_n, n \in I$, on N_+ by the formula:

$$\chi_n(x) = (\chi, x p_{-n} x^{-1}), \qquad x \in N_+. \tag{23}$$

Note that here we consider the action of N_+ on $\widetilde{\mathfrak{g}}$ and the pairing on $\widetilde{\mathfrak{g}}$.

Theorem 5. $\mathbb{C}[N_+] \simeq \mathbb{C}[u_i^{(n)}]_{i=1,\dots,l;n\geq 0} \otimes \mathbb{C}[\chi_n]_{n \in I}$.

Proof. Let us show that the functions $u_i^{(n)}$'s and χ_n's are algebraically independent. In order to do that, let us compute the values of the differentials of these functions at the origin. Those are elements of the cotangent space to the origin, which is isomorphic to the dual space $\mathfrak{n}_+^*$ of $\mathfrak{n}_+$.

It follows from Proposition 5 that $\mathfrak{n}_+^*$ can be written as $\mathfrak{n}_+^* = \mathfrak{a}_+^* \oplus \widetilde{\mathfrak{n}}_+^*$, where $\mathfrak{a}_+^* = \mathrm{Ker}(\mathrm{ad}^* p_{-1})$ and $\widetilde{\mathfrak{n}}_+^*$ is the annihilator of $\mathfrak{a}_+$ with respect to the pairing between $\mathfrak{n}_+$ and $\mathfrak{n}_+^*$. Moreover, if we decompose $\widetilde{\mathfrak{n}}_+^*$ with respect to the principal gradation as $\oplus_{j=1}^\infty \widetilde{\mathfrak{n}}_+^{*,j}$, then $\dim \widetilde{\mathfrak{n}}_+^{*,j} = l$ for all $j > 0$, and $\mathrm{ad}^* p_{-1}$ maps $\widetilde{\mathfrak{n}}_+^{*,j}$ isomorphically to $\widetilde{\mathfrak{n}}_+^{*,j-1}$ for $j > 1$.

By construction of u_i's given in [7], $du_i|_1, i = 1, \dots, l$, form a basis of $\widetilde{\mathfrak{n}}_+^{*,1}$, and hence $du_i^{(n)}|_1, i = 1, \dots, l$, form a basis of $\widetilde{\mathfrak{n}}_+^{*,n}$. Thus, the covectors $du_i^{(n)}|_1, i = 1, \dots, l; n \geq 0$, are linearly independent. Let us show now that the covectors $d\chi_n|_1$ are linearly independent from them and among themselves. For that it is sufficient to show that the pairing between $dF_m|_1$ and p_n is non-zero if and only if $n = m$. But we have:

$$p_n^R \cdot (\chi, xp_{-m}x^{-1}) = (\chi, x[p_n, p_{-m}]x^{-1}) \tag{24}$$

$$= (\chi, n(p_n, p_{-n})C)\delta_{n,-m} = n(p_n, p_{-n})(\chi, C)\delta_{n,-m},$$

where h is the Coxeter number. Therefore this pairing equals $n(p_n, p_{-n})(\chi, C)\delta_{n,-m}$. This satisfies the condition above.

Thus, the functions $u_i^{(n)}$'s and χ_n's are algebraically independent. Hence we have an embedding $\mathbb{C}[u_i^{(n)}]_{i=1,\dots,l;n\geq 0} \otimes \mathbb{C}[\chi_n]_{n\in I} \to \mathbb{C}[N_+]$. But the characters of the two spaces with respect to the principal gradation are both equal to

$$\prod_{n\geq 0}(1 - q^n)^{-l} \prod_{i\in I}(1 - q^i)^{-1}.$$

Hence this embedding is an isomorphism. $\square$

4.2. Another proof of Theorem 3. Now we will explain another point of view on the equivalence between the formalisms of [3] and [7] established in Sect. 3.

In any finite-dimensional representation of N_+, each element x of N_+ is represented by a matrix whose entries are Taylor series in t with coefficients in $\mathbb{C}[N_+]$.

Denote $c_n = (n(p_n, p_{-n})(\chi, C))^{-1}$.

Proposition 6. *Let x be an element of N_+. We associate to it another element of N_+,*

$$\overline{x} = x \exp\left(-\sum_{n\in I} c_n p_n \chi_n(x)\right)$$

In any finite-dimensional representation of N_+, $\overline{x}$ is represented by a matrix whose entries are Taylor series with coefficients in the ring of differential polynomials in $u_i, i = 1, \dots, l$

The map $N_+ \to N_+$ which sends x to $\overline{x}$ is constant on the right A_+–cosets, and hence defines a section $N_+/A_+ \to N_+$.

Proof. Each entry of

$$\overline{x} = x \exp\left(-\sum_{n\in I} c_n p_n \chi_n(x)\right)$$

is a function on N_+. According to Theorem 1 and Theorem 5, to prove the proposition it is sufficient to show that each entry of $\overline{x}$ is invariant under the right action of $\mathfrak{a}_+$. By formula (24) we obtain for each $m \in I$:

$$p_m^R \cdot \chi_n = c_n^{-1} \delta_{n,-m},$$

and hence

$$p_m^R \left(x \exp \left(- \sum_{n \in I} c_n p_n \chi_n(x) \right) \right) =$$

$$x p_m \exp \left(- \sum_{n \in I} c_n p_n \chi_n \right) + x \exp \left(- \sum_{n \in I} c_n p_n \chi_n \right) (-p_m) = 0.$$

Therefore $\overline{x}$ is right $\mathfrak{a}_+$–invariant.

To prove the second statement, let a be an element of A_+ and let us show that $\overline{xa} = \overline{x}$. We can write: $a = \exp\left(\sum_{i \in I} \alpha_i p_i\right)$. Then according to formulas (23) and (24), $\chi_n(xa) = (\chi, xap_{-n}a^{-1}x^{-1}) = (\chi, xp_{-n}x^{-1}) + c_n^{-1}\alpha_n = \chi_n(x) + c_n^{-1}\alpha_n$. Therefore

$$\overline{xa} = xa \exp \left(- \sum_{n \in I} \alpha_n p_n - \sum_{n \in I} c_n p_n \chi_n(x) \right) = \overline{x}.$$

□

Consider now the matrix $\overline{x}$. According to Proposition 6, the entries of $\overline{x}$ are Taylor series with coefficients in differential polynomials in u_i's. Hence we can apply to $\overline{x}$ any derivation of $\mathbb{C}[u_i^{(n)}]$, in particular, $\partial_n = p_{-n}^R$. In the following proposition we consider p_i and $\mathbf{u}$ as matrices acting in a finite-dimensional representation.

Lemma 4. *In any finite-dimensional representation of N_+, the matrix of $\overline{x}$ satisfies:*

$$\overline{x}^{-1}(\partial_n + (\overline{x}p_{-n}\overline{x}^{-1})_-)\overline{x} = \partial_n + p_{-n} - \sum_{i \in I} c_i (p_{-n}^R \cdot \chi_i) p_i. \tag{25}$$

Proof. Using formula (5), we obtain:

$$\begin{aligned} \overline{x}^{-1}(\partial_n + (\overline{x}p_{-n}\overline{x}^{-1})_-)\overline{x} &= \partial_n + \overline{x}^{-1}(p_{-n}^R \overline{x}) + \overline{x}^{-1}(\overline{x}p_{-n}\overline{x}^{-1})_- \overline{x} \\ &= \partial_n + \overline{x}^{-1}(\overline{x}p_{-n}\overline{x}^{-1})_+ \overline{x} - \sum_{i \in I} c_i (p_{-n}^R \cdot \chi_i) p_i + \overline{x}^{-1}(\overline{x}p_{-n}\overline{x}^{-1})_- \overline{x} \\ &= \partial_n + p_{-n} - \sum_{i \in I} c_i (p_{-n}^R \cdot \chi_i) p_i, \end{aligned}$$

which coincides with (25). □

Second proof of Theorem 3. Let $K(z)$ be the map $\mathbb{A}^1 \to N_+/A_+$ assigned to a smooth function $\mathbf{u} : \mathbb{A}^1 \to \mathfrak{h}$ by Theorem 1. Let $\overline{K}(z)$ be the element of N_+ corresponding to $K(z)$ under the map $N_+/A_+ \to N_+$ defined in Proposition 6. By construction, $\overline{K}(z)$ lies in the A_+–coset of $K(z)$.

According to Lemma 2, $(\overline{K}(z)p_{-1}\overline{K}(z)^{-1})_- = p_{-1} + \mathbf{u}(z)$. Setting $n = 1$ in formula (25) we obtain:

$$\overline{K}^{-1}(\partial_z + p_{-1} + \mathbf{u}(z))\overline{K} = \partial_z + p_{-1} - \sum_{i\in I} c_i(p^R_{-1}\cdot\chi_i)p_i.$$

This shows that $\overline{K}(z)$ gives a solution to equation (15), and hence lies in the A_+–coset of the Drinfeld-Sokolov dressing operator $M(z)$. Therefore the cosets of $K(z)$ and $M(z)$ coincide. □

It is possible to lift the map $\mathbf{u}(z) \to N_+/A_+$ constructed in [3] and [7] to a map $\mathbf{u}(z) \to N_+$. We can first attach to $\mathbf{u}(z)$ the coset $K(z)$ and then an element $\overline{K}(z)$ of N_+ defined as in the proof of Theorem 3. In the next section we will show that $H_n = p^R_{-1}\cdot\chi_n \in \mathbb{C}[u_i^{(n)}] \subset \mathbb{C}[N_+]$ (recall that $\chi_n \not\in \mathbb{C}[u_i^{(n)}]$). Since $p^R_{-1} \equiv \partial_z$, we can view χ_n as $\int_{-\infty}^z H_n dz$. Hence we can construct the image of $\mathbf{u}(z)$ in N_+ by the formula

$$\widetilde{K}(z) = \overline{K}(z)\exp\left(\sum_{n\in I} c_n p_n \int_{-\infty}^z H_n dz\right).$$

Comparing (25) and (15), we can write an equivalent formula

$$M(z)\exp\left(-\sum_{n\in I} p_n \int_{-\infty}^z h_n(z)dz\right),$$

where $h_n(z)$ are defined by formula (15). Note that the last formula does not depend on the choice of $M(z)$.

We see that in contrast to the map to N_+/A_+, which is local, i.e. depends only on the jet of $\mathbf{u}(z)$ at z, the map to N_+ is non-local.

Remark 8. In [3] it was proved that h_n is the hamiltonian of the nth equation of the mKdV hierarchy. In the next section we will prove in a different way that $p^R_{-1}\cdot\chi_n$ is proportional to the hamiltonian of the nth equation of the mKdV hierarchy. □

Remark 9. Now we can write an explicit formula for the Baker-Akhiezer function associated to $\mathbf{u}$. Recall that this function is a formal solution of equations (21). From formula (25) we obtain the following solution:

$$\begin{aligned}\Psi(\mathbf{t}) &= \overline{K}(\mathbf{t})\exp\left(-\sum_{i\in I} p_{-i}t_i - \sum_{i\in I} c_i p_i \int_{-\infty}^z H_i(z)dz\right)\\ &= \widetilde{K}(\mathbf{t})\exp\left(-\sum_{i\in I} p_{-i}t_i\right),\end{aligned}$$

where $\mathbf{t} = \{t_i\}_{i\in I}$ and t_i's are the times of the hierarchy (in particular, $t_1 = z$). On the other hand, by construction, the action of the vector field ∂_n of the mKdV

hierarchy on $\widetilde{K} \in N_+$ corresponds to the right action of $p_{-n} \in \mathfrak{a}_-$ on $N_+ \subset B_- \backslash G$. Hence if $\widetilde{K}_0 \in N_+$ is the initial value of $\widetilde{K}$, when all $t_i = 0$, then

$$\widetilde{K}(\mathbf{t}) = \left(\widetilde{K}_0 \Gamma(\{t_i\})\right)_+ ,$$

where

$$\Gamma(\mathbf{t}) = \exp\left(\sum_{i \in I} p_{-i} t_i\right)$$

and g_+ denotes the projection of $g \in B_- \cdot N_+ \subset G$ on N_+ (it is well-defined for almost all t_i's). Finally, we obtain:

$$\Psi(\mathbf{t}) = \left(\widetilde{K}_0 \Gamma(\mathbf{t})\right)_+ \Gamma(\mathbf{t})^{-1}.$$

Note that this formula differs slightly from the one given in [16, 9] because in those papers another realization of the flag manifold was chosen: G/B_- instead of our $B_- \backslash G$. $\square$

5. One-cocycles, hamiltonians and τ-functions.

In the previous section we established the equivalence between the approaches of [3] and [7] to the mKdV hierarchies. In both papers the mKdV equations were proved to be hamiltonian. In this section we will discuss explicit formulas for the hamiltonians of the mKdV equation and for some closely related cohomology classes of $\mathfrak{n}_+$. Note also that both in [3] and [7] it was shown that the hamiltonians of the mKdV equations are integrals of motion of the corresponding affine Toda equation (see also [13, 15]).

5.1. Connection between the hamiltonians and the $\mathfrak{n}_+$–cohomology. In [6, 7] the space spanned by the hamiltonians of the mKdV equations was identified with the first cohomology of $\mathfrak{n}_+$ with coefficients in π_0, $H^1(\mathfrak{n}_+, \pi_0)$. Let us briefly recall how to assign an mKdV hamiltonian to a cohomology class.

The cohomology of $\mathfrak{n}_+$ with coefficients in π_0 can be computed using the Koszul complex $\pi_0 \otimes \bigwedge^*(\mathfrak{n}_+^*)$. A cohomology class from $H^1(\mathfrak{n}_+, \pi_0)$ is represented in the Koszul complex by a functional f on $\mathfrak{n}_+$ with coefficients in π_0, which satisfies the cocycle condition

$$f([a, b]) - a \cdot f(b) + b \cdot f(a) = 0.$$

This condition uniquely determines f by its values $f_i \in \pi_0$ on the generators $e_i, i = 0, \ldots, l$, of $\mathfrak{n}_+$. Now set $g_i = \partial_z f_i - u_i f_i, i = 0, \ldots, l$. As shown in [7], there exists $h \in \pi_0$, such that $g_i = e_i^L \cdot h, i = 0, \ldots, l$.

It was proved in [6, 7] that $H^1(\mathfrak{n}_+, \pi_0) \simeq \mathfrak{a}_+^*$. Using the invariant inner product, we can identify $\mathfrak{a}_+^*$ with $\mathfrak{a}_-$. Let f_n be the cohomology class corresponding to $p_{-n} \in \mathfrak{a}_-$. Then $h_n \in \pi_0$ constructed from f_n is, by definition, the density of the hamiltonian of

the nth mKdV equation (i.e. the projection of h_n onto the space of local functionals $\pi_0/(\operatorname{Im}\partial_z \oplus \mathbb{C})$ is an mKdV hamiltonian).

Below we give explicit formulas for $f_n(e_i)$ and h_n as functions on N_+/A_+. To simplify notation we will simply write e_i for e_i^L and p_{-n} for p_{-n}^R.

5.2. Formulas for one-cocycles. Now recall that $\pi_0 \simeq \mathbb{C}[N_+/A_+]$. Hence the values of a one-cocycle of $\mathfrak{n}_+$ with coefficients in π_0 can be viewed as a regular function on N_+/A_+.

Proposition 7. *There exists a one-cocycle ϕ_n such that*

$$(\phi_n(e_i))\,(K) = (e_i, K p_{-n} K^{-1}), \qquad K \in N_+/A_+. \tag{26}$$

The cohomology classes corresponding to these cocycles span $H^1(\mathfrak{n}_+, \pi_0)$. In particular, if n is a multiplicity free exponent, then the cohomology classes defined by ϕ_n and f_n coincide up to a constant multiple.

Proof. There exists a unique element $\rho^\vee \in \widetilde{\mathfrak{h}} \simeq \widetilde{\mathfrak{h}}^*$, such that $(\alpha_i, \rho^\vee) = 1, \forall i = 0, \ldots, l$, and $(d, \rho^\vee) = 0$. But $\widetilde{\mathfrak{h}}^*$ is isomorphic to $\widetilde{\mathfrak{h}}$ via the non-degenerate inner product $(\cdot,\cdot)$. Let us use the same notation for the image of $\rho^\vee$ in $\widetilde{\mathfrak{h}}$ under this isomorphism. Then $\rho^\vee$ satisfies: $[\rho^\vee, e_i] = e_i, [\rho^\vee, h_i] = 0, [\rho^\vee, f_i] = -f_i, i = 0, \ldots, l$. Thus, the adjoint action of $\rho^\vee$ on $\widetilde{\mathfrak{g}}$ coincides with the action of the principal gradation.

Any function $F \in \mathbb{C}[N_+]$ can be viewed as an element of the zeroth group of the Koszul complex of the cohomology of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+]$. The coboundary of this element is a (trivial) one-cocycle, whose value on e_i is $e_i \cdot F \in \mathbb{C}[N_+], i = 0, \ldots, l$.

Consider a function $F_n = \rho_n^\vee$ on N_+ defined by the formula

$$F_n(x) = (\rho^\vee, x p_{-n} x^{-1}) \tag{27}$$

Note that here p_{-n} is considered as an element of $\widetilde{\mathfrak{g}}$ and we consider the adjoint action of N_+ on $\widetilde{\mathfrak{g}}$. The value of the corresponding one-cocycle on e_i is equal to $e_i \cdot F_n$. We have:

$$(e_i \cdot F_n)(x) = (\rho^\vee, [e_i, x p_{-n} x^{-1}]) = ([\rho^\vee, e_i], x p_{-n} x^{-1}) = (e_i, x p_{-n} x^{-1}).$$

Thus, there exists a one-cocycle f of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+]$, such that

$$f(e_i) = (e_i, x p_{-n} x^{-1}), i = 0, \ldots, l. \tag{28}$$

Moreover, $f(e_i)$ is A_+–invariant for all $i = 0, \ldots, l$. Indeed,

$$\begin{aligned}(p_m \cdot f(e_i))(x) &= (e_i, x[p_m, p_{-n}]x^{-1}) = n(p_n, p_{-n})(e_i, x C x^{-1})\delta_{n,-m} \\ &= n(p_n, p_{-n})(e_i, C)\delta_{n,-m} = 0.\end{aligned}$$

Therefore formula (28) defines a one-cocycle of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+/A_+] \simeq \pi_0$. This is the cocycle ϕ_n.

By construction, ϕ_n is a trivial one-cocycle of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+]$. But it is non-trivial as a one-cocycle of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+/A_+]$. Indeed, if it were a coboundary, there would exist an A_+–invariant function $\widetilde{F}_n$ on N_+, such that $\phi_n(e_i) = e_i \cdot \widetilde{F}_n$. But then $e_i \cdot (\widetilde{F}_n - F_n) = 0$ for all i, and $\widetilde{F}_n - F_n$ is N_+–invariant, and hence constant. However, by (24), $p_n \cdot F_n = nh(p_n, p_{-n}) \neq 0$, where h is the Coxeter number of $\widetilde{\mathfrak{g}}$. Hence the function F_n is not A_+–invariant.

Thus, $\widetilde{F}_n - F_n$ can not be a constant function. Therefore ϕ_n defines a non-zero cohomology class. Let us compute its degree with respect to the principal gradation. We have:

$$\left(\rho^\vee \cdot (\phi_n(e_i))\right)(x) = (e_i, [(x\rho^\vee x^{-1})_+, xp_{-n}x^{-1}])$$

$$= (e_i, [x\rho^\vee x^{-1}, xp_{-n}x^{-1}]) - (e_i, [(x\rho^\vee x^{-1})_-, xp_{-n}x^{-1}])$$

$$= (e_i, x[\rho^\vee, p_{-n}]x^{-1}) - ([e_i, \rho^\vee], xp_{-n}x^{-1}]) = (-n+1)\phi_n(e_i).$$

Hence the degree of ϕ_n equals $-n$.

For multiplicity free exponent n this implies that the cohomology class of ϕ_n is proportional to that of f_n. For the multiple exponents i which occur in the case of $D^{(1)}_{2n}$ (see Remark 1), we need to show that the cocycles ϕ^1_i and ϕ^2_i, corresponding to two linearly independent elements p^1_{-i} and p^2_{-i} of $\mathfrak{a}_-$ of degree $-i$, are linearly independent. But a linear combination $\alpha\phi^1_i + \beta\phi^2_i$ of these cocycles is just the cocycle corresponding to $\alpha p^1_{-i} + \beta p^2_{-i} \in \mathfrak{a}_-$. The argument that we used above can be applied to the cocycle $\alpha\phi^1_i + \beta\phi^2_i$ to show that it is non-trivial unless both α and β equal 0. $\square$

Remark 10. Homogeneous functions on N_+/A_+ are necessarily algebraic. Thus, $\phi_n(e_i) \in \mathbb{C}[N_+/A_+]$. $\square$

Remark 11. One can show in the same way as above that for any $\chi \in \widetilde{\mathfrak{h}}$, such that $(\chi, C) \neq 0$, there exists a one-cocycle $\widetilde{\chi}_n$ of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[N_+/A_+]$, which satisfies the following property: considered as an A_+–invariant function on N_+, $\widetilde{\chi}_n(e_i)$ equals $e_i \cdot \chi_n$, where χ_n is the function on N_+ defined in Sect. 4.1. The one-cocycle $\widetilde{\chi}_n$ is homologous to F_n, suitably normalized. $\square$

5.3. Formulas for hamiltonians. Now we can find a formula for the density of the nth mKdV hamiltonian using Proposition 7 and the procedure of Sect. 5.1.

Proposition 8. *The function H_n on N_+/A_+, such that*

$$H_n(K) = (p_{-1}, Kp_{-n}K^{-1}), \qquad K \in N_+/A_+$$

is a density of the nth hamiltonian of the mKdV hierarchy.

Proof. We have to show that

$$e_i \cdot H_n = p_{-1}\phi_n(e_i) - u_i\phi_n(e_i), \qquad i = 0, \dots, l. \tag{29}$$

Let us consider functions on N_+/A_+ as A_+–invariant functions on N_+. Recall from Sect. 2 that there is a unique up to a constant isomorphism ϵ_λ between $\mathbb{C}[N_+]$ and the contragradient Verma module M_λ^*, which commutes with the left action of $\mathfrak{n}_+$. For $a \in \mathfrak{g}$ the operator $\epsilon_\lambda a \epsilon_\lambda^{-1}$ on $\mathbb{C}[N_+]$ is the first order differential operator $a^R + f_\lambda(a)$. Here $f_\lambda(a) = \epsilon_\lambda^{-1}(a \cdot v_\lambda)$ We know that $u_i = f_{\alpha_i}(p_{-1})$, see [7] and Sect. 2. Hence $\epsilon_{-\alpha_i}^{-1} p_{-1} \epsilon_{-\alpha_i} = p_{-1}^R - u_i$, and hence formula (29) can be rewritten as

$$\epsilon_{-\alpha_i}(e_i \cdot H_n) = p_{-1} \cdot \epsilon_{-\alpha_i}(\phi_n(e_i)) \qquad i = 0, \dots, l. \tag{30}$$

Let us show that $H_n = p_{-1} \cdot F_n$, where the function $F_n \in \mathbb{C}[N_+]$ is defined by formula (27). Indeed,

$$\begin{aligned}(p_{-1} \cdot F_n)(x) &= (\rho^\vee, [(xp_{-1}x^{-1})_+, xp_{-n}x^{-1}]) = -(\rho^\vee, [(xp_{-1}x^{-1})_-, xp_{-n}x^{-1}]) \\ &= -([\rho^\vee, (xp_{-1}x^{-1})_-], xp_{-n}x^{-1}) = ([\rho^\vee, p_{-1}], xp_{-n}x^{-1}) = (p_{-1}, xp_{-n}x^{-1}).\end{aligned}$$

The fact that H_n is A_+–invariant can be proved in the same way as for $\phi_n(e_i)$.

Now recall that the map $\epsilon_{-\alpha_i} e_i : \mathbb{C}[N_+] \to M_{-\alpha_i}^*$ commutes with the action of $\mathfrak{g}$, where $\mathfrak{g}$ acts on $\mathbb{C}[N_+]$ from the right by vector fields, see [7], Sect. 4. Therefore we obtain

$$\epsilon_{-\alpha_i} e_i (p_{-1} \cdot F_n) = p_{-1} \cdot \epsilon_{-\alpha_i} e_i(F_n).$$

This implies formula (30) if we take into account that $H_n = p_{-1} \cdot F_n$ and $\phi_n(e_i) = e_i \cdot F_n$. □

Remark 12. Our formula for the hamiltonians is equivalent to the formula given by Wilson [15], (4.10). □

Remark 13. One can also construct the density of the nth hamiltonian as $p_{-1} \cdot \chi_n$ where χ_n was defined in Sect. 4.1. For different χ, these densities, suitably normalized, differ by total derivatives, and hence define the same hamiltonian. □

5.4. Involutivity of the hamiltonians. Now we want to prove that the Poisson bracket between two mKdV hamiltonians vanishes. This is equivalent to showing that $p_{-n} \cdot H_m = p_{-1} H_{n,m}$ for some $H_{n,m} \in \mathbb{C}[N_+/A_+]$ (see [7]).

Proposition 9. *Define* $H_{n,m} \in \mathbb{C}[N_+/A_+]$ *by formula*

$$H_{n,m}(K) = -([\rho^\vee, (Kp_{-n}K^{-1})_-], Kp_{-m}K^{-1}).$$

Then

$$p_{-n} \cdot H_m = p_{-m} \cdot H_n = p_{-1} \cdot H_{n,m}.$$

Proof. We have:

$$(p_{-n} \cdot H_m)(K) = (p_{-1}, [(Kp_{-n}K^{-1})_+, Kp_{-m}K^{-1}]) =$$
$$(p_{-1}, [(Kp_{-n}K^{-1})_+, (Kp_{-m}K^{-1})_-]),$$

because $(p_{-1}, [y_1, y_2]) = 0$ if $y_1, y_2 \in \mathfrak{n}_+$. On the other hand,

$$(p_{-m} \cdot H_n)(K) = (p_{-1}, [(Kp_{-m}K^{-1})_+, Kp_{-n}K^{-1}]) =$$
$$-(p_{-1}, [(Kp_{-m}K^{-1})_-, Kp_{-n}K^{-1}]) = -(p_{-1}, [(Kp_{-m}K^{-1})_-, (Kp_{-n}K^{-1})_+]),$$

because $(p_{-1}, y) = 0$ if $y \in \mathfrak{b}_-$. Therefore $p_{-n} \cdot H_m = p_{-m} \cdot H_n$.

Consider now H_m as an A_+–invariant function on N_+. Then we have: $H_m = p_{-1} \cdot F_m$. Hence $p_{-n} \cdot H_m = p_{-1} \cdot (p_{-n} \cdot F_m)$. Let $H_{n,m} = p_{-n} \cdot F_m$. We obtain:

$$(p_{-n} \cdot F_m)(x) = (\rho^\vee, [(xp_{-n}x^{-1})_+, xp_{-m}x^{-1}]) = -(\rho^\vee, [(xp_{-n}x^{-1})_-, xp_{-m}x^{-1}])$$
$$= -([\rho^\vee, (xp_{-n}x^{-1})_+], xp_{-m}x^{-1}).$$

The latter expression is A_+–invariant, which can be shown in the same way as in the proof of Proposition 7. Hence $H_{n,m} \in \mathbb{C}[N_+/A_+]$ and $p_{-n} \cdot H_m = p_{-m} \cdot H_n = p_{-1} H_{n,m}$. $\square$

5.5. Connection with τ–functions. The τ–functions have the following meaning from our point of view. For $\lambda \in \widetilde{\mathfrak{h}}^*$, consider the contragradient Verma module M_λ^* over $\mathfrak{g}$. This module can be realized in the space of sections of a line bundle ξ_λ over N_+, considered as a big cell of the flag manifold $B_- \backslash G$. By definition, the τ–function τ_λ corresponding to λ is the unique up to a constant N_+–invariant section of ξ_λ over N_+.

Remark 14. This should be compared with the definition of the τ–functions in the framework of the Grassmannian approach [2, 14, 16]. $\square$

Note that ξ_λ can be trivialized over N_+, and so there exists a unique up to a non-zero constant isomorphism between the space of sections of ξ_λ and $\mathbb{C}[N_+]$. Under this isomorphism, τ_λ corresponds to a constant function on N_+.

Let $\Lambda_i, i = 0, \dots, l$, be the fundamental weights of the affine algebra $\mathfrak{g}$. We call τ_{Λ_i} the ith τ–function of $\mathfrak{g}$ and denote it by τ_i. Let us also set $\tau = \tau_{\rho^\vee}$.

According to Proposition 2, for any $a \in \mathfrak{g}$, $a \cdot \tau_\lambda = f_\lambda(a)\tau_\lambda$, where $f_\lambda(a)(x) = \langle \lambda, xax^{-1} \rangle$.

In particular, we see that $e^{\varphi_i} = \tau_{\alpha_i}$, and $p_{-1}e^{\varphi_i} = \partial_z e^{\varphi_i} = u_i e^{\varphi_i}$. Note that e^{φ_i} can be expressed in terms of τ_j's. For example, for $\mathfrak{g} = \widehat{\mathfrak{sl}}_N$ we have: $e^{\varphi_i} = \tau_{i-1}^{-1}\tau_i^2\tau_{i+1}^{-1}$, which is well-known.

Now we can interpret the functions F_n as logarithmic derivatives of τ. Indeed, we obtain $\partial_n \tau = F_n \tau$, so that we can formally write: $F_n = \partial_n \log \tau$. Further, $H_n = \partial_n \partial_z \log \tau$, and, more generally, $H_{n,m} = \partial_n \partial_m \log \tau$, which coincides with known results. Similarly, we can write: $u_i^{(n)} = \partial_z^{n+1} \log \tau_{\alpha_i}$.

Remark 15. More generally, we have the following formula for the function χ_n defined in Sect. 4.1: $\chi_n = \partial_n \tau_\chi / \tau_\chi$. $\square$

To summarize, the group N_+ has natural coordinates $u_i^{(n)}$ and F_n, which can be obtained as logarithmic derivatives of τ–functions. The vector fields p^R_{-n} written in terms of these coordinates provide the flows of the mKdV hierarchy, and the vector field $\sum_{i=0}^l e_i^L$ written in terms of these coordinates gives the affine Toda equation [7].

5.6. Example of $\widehat{\mathfrak{sl}}_2$. Here we will write explicit formulas for the action of the generators of the nilpotent subalgebra of $\widehat{\mathfrak{sl}}_2$ and mKdV hamiltonians on the corresponding unipotent subgroup.

According to the results of this section, we have an isomorphism

$$\mathbb{C}[N_+] \simeq \mathbb{C}[u^{(n)}, F_m]_{n\geq 0, m\,\mathrm{odd}}.$$

The left action of the generators e_0 and e_1 of $\mathfrak{n}_+$ on $\mathbb{C}[N_+]$ is given by

$$e_0 = -\sum_{n\geq 0} P_n^+ \frac{\partial}{\partial u^{(n)}} + \sum_{m\,\mathrm{odd}} \phi_m(e_0) \frac{\partial}{\partial F_m},$$
$$e_1 = -\sum_{n\geq 0} P_n^- \frac{\partial}{\partial u^{(n)}} + \sum_{m\,\mathrm{odd}} \phi_m(e_1) \frac{\partial}{\partial F_m},$$

where $P_n^\pm$ are elements of $\mathbb{C}[u^{(n)}]$, defined recursively as follows: $P_0^\pm = 1$, $P_{n+1}^\pm = \partial P_n^\pm \pm u P_n^\pm$, and $\phi_m(e_i)$ are the values of a one-cocycle ϕ_m of $\mathfrak{n}_+$ with coefficients in $\mathbb{C}[u_i^{(n)}]$ of degree m.

The right action of p_k, k positive odd, is given by $4k\partial/\partial F_k$, and the action of p_{-k}, k positive odd, is given by

$$p_{-k} = \sum_{n\geq 0} (\partial^{n+1} q_k) \frac{\partial}{\partial u^{(n)}} + \sum_{m\,\mathrm{odd}} H_{k,m} \frac{\partial}{\partial F_m}.$$

The mth mKdV equation now reads:

$$\partial_m u = q_m.$$

This equation is hamiltonian with the hamiltonian $(1/m)H_{m,1}$, and hence

$$q_m = \frac{1}{m} \frac{\delta H_{m,1}}{\delta u}.$$

The involutivity of the hamiltonians means that

$$\sum_{n\geq 0} (\partial^{n+1} q_k) \frac{\partial H_{m,1}}{\partial u^{(n)}} = \partial H_{k,m}$$

(note that $H_{k,m} = H_{m,k}$ and $H_{m,1} = H_{1,m} = H_m$).

The KdV variable is $v = \frac{1}{2}u^2 + u'$, and $\mathbb{C}[v^{(n)}]_{n\geq 0} \subset \mathbb{C}[u^{(n)}]_{n\geq 0}$ coincides with the e_1–invariant subspace of $\mathbb{C}[u^{(n)}]_{n\geq 0}$.

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References

[1] M.F. de Groot, T.J. Hollowood, J.L. Miramontes, Comm. Math. Phys. **145** (1992) 57-84.
[2] E. Date, M. Jimbo, M. Kashiwara, T. Miwa, in Non-linear Integrable Systems – Classical Theory and Quantum Theory, M. Jimbo, T. Miwa (eds.), pp. 39-120, Singapore, World Scientific, 1983.
[3] V.G. Drinfeld, V.V. Sokolov, Sov. Math. Dokl. **23** (1981) 457-462; J. Sov. Math. **30** (1985) 1975-2035.
[4] B. Enriquez, Theor. Math. Phys. **98** (1994), 256-258.
[5] B. Enriquez, B. Feigin, Theor. Math. Phys. **103** (1995), 738-756 (hep-th/9409075).
[6] B. Feigin, E. Frenkel, in Lect. Notes in Math. **1620**, pp. 349-418, Springer Verlag, 1995 (hep-th/9310022).
[7] B. Feigin, E. Frenkel, Invent. Math. **120** (1995) 379-408 (hep-th/9311171).
[8] B. Feigin, E. Frenkel, *Non-linear Schrödinger equations and Wakimoto modules*, to appear.
[9] T. Hollowood, J.L. Miramontes, Comm. Math. Phys. **157** (1993) 99-117.
[10] V.G. Kac, Infinite-dimensional Lie Algebras, 3rd Edition, Cambridge University Press, 1990.
[11] V.G. Kac, Adv. Math. **30** (1978) 85-136.
[12] B. Kostant, in Lect. Notes in Math. **466**, pp. 101-128, Springer Verlag, 1974.
[13] B.A. Kupershmidt, G. Wilson, Comm. Math. Phys. **81** (1981) 189-202.
[14] G. Segal, G. Wilson, Loop groups and equations of KdV type, Publ. Math. IHES **63** (1985) 5-65.
[15] G. Wilson, Ergod. Th. and Dynam. Syst. **1** (1981) 361-380.
[16] G. Wilson, C. R. Acad. Sc. Paris **299**, Serie I (1984) 587-590; Phil. Trans. Royal Soc. London **A 315** (1985) 393-404.
[17] V.E. Zakharov, A.B. Shabat, Funct. Anal. Appl. **13** (1979) 166-174.

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